Prof. Raghuveer Parthasarathy
University of Oregon; Fall 2007
Physics 351

Comments on Analytic Proofs

From grading exercise P2, it seems that some of you are a bit unclear on the meaning and importance of an "analytic proof" of a mathematical relation. Let's consider the following statement: **√2 is a rational number**. Is this true or false? Recall that a rational number is one that can be expressed as a ratio of integers, e.g. 3/4, or 4123/154132. Consider two contradictory claims.

NUMERICAL (COMPUTATIONAL) ARGUMENT THAT √2 IS A RATIONAL NUMBER:

The number √2 is rational if I can find **integers** *p* and *q* such that $\left(p/q\right)^2 = 2$. So, I'll ask my computer to search through billions and billions of possible *p*'s and *q*'s and report on ones that satisfy the above relation. I write a program to do this...

... it finds:

```
p = 2020305089104422;   q = 1428571428571429;
```

Let's check this: I ask the computer to calculate $\left(p/q\right)^2$, or I can do the calculation by hand if I want, and we find that $\left(p/q\right)^2 = 2.000000000000000$. Therefore, **√2 IS a rational number.**

ANALYTIC PROOF THAT √2 IS NOT A RATIONAL NUMBER:

Suppose that √2 is rational. Therefore there exist integers *p* and *q* such that $\left(p/q\right)^2 = 2$, i.e. $p^2 = 2q^2$. If *p* and *q* are both even numbers, we can divide each by 2, and keep dividing by 2, until we find a "new" p and q that still satisfy $p/q = \sqrt{2}$, but that are **not** both even. So: $p^2 = 2q^2$, and *p* and *q* are not both even. Since p^2 is a multiple of 2, in other words it's an even number, *p* must be even, since there's no way we can square an odd number to get an even one (e.g. 3×3=9...). So *p* is even. So we can write $p = 2w$, where *w* is some other integer. Our relation $p^2 = 2q^2$ becomes $4w^2 = 2q^2$, or $2w^2 = q^2$, which, just as we did for *p* before, tells us that *q* is an even number. Therefore *p* and *q* are both even. But: this contradicts our earlier claim that *p* and *q* are not both even! Therefore our premise must be wrong, and so **there cannot exist integers *p* and *q*** such that $\left(p/q\right)^2 = 2$, and so **√2 is irrational**.

Which claim do you believe?

HISTORY: The existence of irrational numbers was discovered in ancient Greece and ancient India over 2500 years ago. Legend has it that a Pythagorean (a follower of Pythagoras) discovered this property of √2, which contradicted the mystical & mathematical beliefs of the Pythagoreans, which held that all numbers are expressible as rational fractions, and so they tossed him off a ship, murdering him.